

Using the Shephard's Lemma to obtain Demand Functions

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Hicksian Demand Function and Shepard's Lemma.

- Minimise expenditure subject to a constant utility level:

$$\min_{x,y} [p_x \cdot x + p_y \cdot y] \quad \text{s.t.} \quad u(x,y) = \bar{u}.$$

Hicksian Demand Function

Hicksian demand function is the compensated demand function that keeps utility level constant and thus only measures the substitution effect. It is the solution to the following problem where the expenditure $p_x \cdot x + p_y \cdot y$ is minimised subject to a particular utility level \bar{u} .

$$\mathcal{L} = [p_x \cdot x + p_y \cdot y] + \lambda [\bar{u} - u(x,y)]$$

The solution to the above problem is the following Hicksian Demand function

$$x^h = x^h(\bar{u}, p_x, p_y)$$

$$y^h = y^h(\bar{u}, p_x, p_y)$$

Expenditure function

We can use these Hicksian Demand function to construct the expenditure function.

$$E = p_x \cdot x^h + p_y \cdot y^h$$

substituting the Hicksian demand function from above

$$E(\bar{u}, p_x, p_y) = p_x \cdot x^h(\bar{u}, p_x, p_y) + p_y \cdot y^h(\bar{u}, p_x, p_y)$$

Interpretation minimum expenditure required to produce utility \bar{u} .

Shephard's Lemma is an easy way to get Hicksian Demand function from expenditure function:

$$\frac{\partial E}{\partial p_x} = x^h$$
$$\frac{\partial E}{\partial p_y} = y^h$$

You have remember that this is actually more complicated than it looks above because it should be written as

a complicated version

$$\frac{\partial E(\bar{u}, p_x, p_y)}{\partial p_x} = x^h(\bar{u}, p_x, p_y)$$

$$\frac{\partial E(\bar{u}, p_x, p_y)}{\partial p_y} = y^h(\bar{u}, p_x, p_y)$$

Exploring the Shephard's Lemma further

It is useful to think about how we derive the Shephard's Lemma especially because it is an excellent application of the envelope theorem.

$$\begin{aligned} \mathcal{L} \Big|_{x=x^h, y=y^h} &= p_x x^h + p_y y^h + \lambda [\bar{u} - u(x^h, y^h)] \\ &= p_x x^h + p_y y^h \\ &= E(\bar{u}, p_x, p_y) \end{aligned}$$

Envelope Theorem

This is because if $\bar{u} - u(x^h, y^h) = 0$. Since x^h and y^h are the solution to the problem, this would obviously imply that $\bar{u} - u(x^h, y^h) = 0$. This trick is known as the envelope theorem where a part of the problem disappears when it is evaluated at the solution.

$$\frac{\partial \mathcal{L}}{\partial p_x} \Big|_{x=x^h, y=y^h} = x^h = \frac{\partial E}{\partial p_x}$$

$$\frac{\partial \mathcal{L}}{\partial p_y} \Big|_{x=x^h, y=y^h} = y^h = \frac{\partial E}{\partial p_y}$$

Obtaining the Demand Functions from Cost Functions.

- Minimise cost subject to a level of production:

$$\min_{k,l} [r \cdot k + w \cdot l] \quad \text{s.t.} \quad f(k, l) = \bar{y}$$

$$\mathcal{L} = [r \cdot k + w \cdot l] + \lambda [\bar{y} - f(k, l)]$$

the solution to the above is the following demand functions for k and l .

$$k^h = k^h(\bar{y}, w, r)$$

$$l^h = l^h(\bar{y}, w, r)$$

This gives us the following cost function.

$$C(\bar{y}, r, w) = r \cdot k^h + w \cdot l^h$$

We can obtain the demand functions from the cost function by just differentiating the cost function $c(\bar{y}, r, w)$

$$\left. \frac{\partial \mathcal{L}}{\partial r} \right|_{k=k^h, l=l^h} = k^h = \frac{\partial C}{\partial r}$$

$$\left. \frac{\partial \mathcal{L}}{\partial w} \right|_{k=k^h, l=l^h} = l^h = \frac{\partial C}{\partial w}$$

Cost Function Remember that the Lagrangian evaluated at the solution, i.e., $k = k^h$ and $l = l^h$, gives you the cost function. This is because $\bar{y} - f(k^h, l^h) = 0$. The cost function gives you the most inexpensive way of producing the output \bar{y} .

Obtaining Demand Function Differentiating the cost function is just an easy way to get the demand function.

Example: 2009, Question 11.

Suppose that a firm has two different plants. The cost functions for these two plants are

$$c_1(w, r, y) = \frac{1}{2} \bar{y}(w + r)$$

$$c_2(w, r, y) = 2\bar{y}\sqrt{wr}$$

where y is output, w is the cost of labour l and r is the cost of capital k .

- Cost function 1:

$$c_1(w, r, \bar{y}) = \frac{1}{2} (w + r)$$

$$\frac{\partial c_1}{\partial r} = \frac{1}{2} \bar{y} = k^h$$

$$\frac{\partial c_1}{\partial w} = \frac{1}{2} \bar{y} = l^h$$

Important to note The above expressions just give us the demand for capital and labour. Since w and r taken as a given value, it implies that the supply for capital and labour is elastic at these factor prices. With an elastic supply, capital and labour are demand determined and give us the following:

$$k = h^h$$

$$l = l^h$$

This gives us the following:

$$\bar{y} = 2k = 2l$$

This implies that k and l are complementary and the output is constrained by whichever factor is scarce. The technology or the production function is given by

$$y = \min [2k, 2l]$$

- Cost function 2:

$$c_2(w, r, y) = 2\bar{y}\sqrt{wr}$$

$$\frac{\partial c_1}{\partial r} = \bar{y} \sqrt{\frac{w}{r}} = k^h = k$$

$$\frac{\partial c_1}{\partial w} = \bar{y} \frac{1}{\sqrt{\frac{w}{r}}} = l^h = l$$

this can be written down as

$$\sqrt{\frac{w}{r}} = \frac{k}{\bar{y}} = \frac{\bar{y}}{l}$$

which gives us the following technology or the production function.

$$\bar{y} = \sqrt{kl}$$

- (a) *Write down the corresponding technologies.*
- Technology associated with cost function 1:
- $$y = \min [2k, 2l]$$
- Technology associated with cost function 2:
- $$\bar{y} = \sqrt{kl}$$
- (b) *How much labour and capital does a cost-minimising firm need to produce one unit in each plant?*
- To produce 1 unit of y using technology 1, you need $k = \frac{1}{2}$ and $l = \frac{1}{2}$.
 - To produce 1 unit of y using technology 2 is given, the firm need to use capital and labour to ensure that $\sqrt{k} = \frac{1}{\sqrt{L}}$ holds. For example, if the firm uses $k = \frac{1}{2}$, it needs to use $l = 2$.
- (c) *Will a cost minimising firm shut down one plant.*
- From the cost functions above, we know the cost of producing \bar{y} units of output in the two plants. The firm will shut down whichever plant is more costly.

- The firm will shut down plant 1 if the following condition holds.

$$c_1(w, r, \bar{y}) > c_1(w, r, \bar{y})$$

$$\frac{1}{2}\bar{y}(w+r) > 2\bar{y}\sqrt{wr}$$

$$\frac{1}{\sqrt{r}} > 4 - \frac{1}{\sqrt{w}}$$

The firm will shut down firm 2 if the inequality is the other way around.