1. a) Production function has constant returns to scale (CRS) if increasing all factors by an equal proportion increases the output by the same proportion.

   Mathematically, a production function has a constant returns to scale if $\lambda Y = F(\lambda K, \lambda L)$ for any $\lambda > 0$. For example if we double the the amounts of capital and labour ($\lambda = 2$), then output also doubles.

   Checking if the production function has CRS.

$$F(\lambda K, \lambda L) = (\lambda K)^{\frac{1}{2}} (\lambda L)^{\frac{1}{2}} = \lambda (K)^{\frac{1}{2}} (L)^{\frac{1}{2}} = \lambda Y$$

   Therefore, production function $Y = K^{\frac{1}{2}} L^{\frac{1}{2}}$ has a constant returns to scale.

   b) To find the per worker production function, divide the production function $Y = K^{\frac{1}{2}} L^{\frac{1}{2}}$ by L:

$$\frac{Y}{L} = \frac{K^{\frac{1}{2}} L^{\frac{1}{2}}}{L}$$

$$y = \frac{K^{\frac{1}{2}}}{L^{\frac{1}{2}}}$$

$$y = \sqrt{k} \qquad \text{where } k = \frac{K}{L} \text{ and } y = \frac{Y}{L}$$

   c) We know the following about country A and B:

      $\delta = 0.05$ (depreciation rate)
      $s_a = 0.1$ (saving rate of country A)
      $s_a = 0.2$ (saving rate of country B)
      $y = \sqrt{k}$ (the production function derived in part (b) for countries A and B)

   The growth rate of capital stock Δk equals the amount of investment $sf(k)$, less the amount of depreciation δk. That is

$$\Delta k = sf(k) - \delta k.$$

In steady state, the capital stock does not grow, so we can write this as

$$sf(k) = \delta k.$$

To find the steady state level of capital per worker, plug the per-worker production into the steady state investment condition and solve for k^*:

$$s\sqrt{k} = \delta k$$
$$\sqrt{k} = \frac{s}{\delta}$$
$$k = \left[\frac{s}{\delta}\right]^2$$

To find the steady-state level of capital per worker k^*, plug the saving rate for each country in the above formula:

Country A: $k_a^* = \left(\frac{s_a}{\delta}\right)^2 = \left(\frac{0.1}{0.05}\right)^2 = 4$

Country B: $k_b^* = \left(\frac{s_b}{\delta}\right)^2 = \left(\frac{0.2}{0.05}\right)^2 = 16$

Now that we have found the k^* for each country, we can calculate the stationary state levels of income per worker for countries A and B because we know that $y = \sqrt{k}$:

$$y_a^* = (4)^{\frac{1}{2}} = 2$$
$$y_b^* = (16)^{\frac{1}{2}} = 4$$

We know that out of each dollar of income, the workers save a fraction s and consume a fraction $(1-s)$. That is, the consumption function $c = (1-s)y$. Since we know the steady state levels of income in the two countries, we find that

Country A: $c_a^* = (1 - s_a)y_a^* = (1 - 0.1) \cdot 2 = 1.8$

Country B: $c_b^* = (1 - s_b)y_b^* = (1 - 0.2) \cdot 4 = 3.2$

d) Using the following facts and equations we calculate the income per worker y, consumption per worker c, and capital per worker k:

$$s_a = 0.1$$
$$s_b = 0.2$$
$$\delta = 0.05$$
$$k_0 = 2 \text{ for both countries}$$
$$y = \sqrt{k}$$
$$c = (1-s)y$$

Table 1: Country A

Year	k	$y = \sqrt{k}$	$c = (1-s_a)y$	$i = sy$	δk	$\delta i - \delta k$
1	2.000	1.414	1.273	0.141	0.100	0.041
2	2.041	1.429	1.286	0.143	0.102	0.041
3	2.082	1.443	1.299	0.144	0.104	0.040
4	2.122	1.457	1.311	0.146	0.106	0.040
5	2.102	1.470	1.323	0.147	0.108	0.039

Table 2: Country B

Year	k	$y = \sqrt{k}$	$c = (1-s_b)y$	$i = sy$	δk	$\delta i - \delta k$
1	2.000	1.414	1.131	0.283	0.100	0.183
2	2.183	1.477	1.182	0.295	0.109	0.186
3	2.369	1.539	1.231	0.308	0.118	0.190
4	2.559	1.600	1.280	0.320	0.128	0.192
5	2.751	1.659	1.327	0.332	0.138	0.194

Note that it take 5 years before the consumption in country B is higher than consumption in country A.

2. Start with the production function $Y = F(K, AL)$ and use the facts that it has constant returns to scale and we multiply the input by a factor $\lambda = \frac{1}{AL}$ to get

$$\tilde{y} = f(\tilde{k})$$

where we define $\tilde{k} = \frac{K}{AL}$ and $\tilde{y} = \frac{Y}{AL}$

$$\frac{Y}{AL} = F\left(\frac{K}{AL}, 1\right)$$

$$\Rightarrow Y = AL \cdot F\left(\frac{K}{AL}, 1\right) = AL \cdot f(\tilde{k})$$

The cost of capital is defined as $r + \delta$. Then, in a competitive equilibrium, the marginal product of capital MPK must be equal to the cost of capital:

$$\begin{aligned}
\text{MPK } &= \frac{\partial Y}{\partial K} = r + \delta \\
\Rightarrow \quad r &= \frac{\partial Y}{\partial K} - \delta = AL \cdot \frac{1}{AL} \cdot F_1\left(\frac{K}{AL}, 1\right) - \delta \\
&= f'(\tilde{k})
\end{aligned}$$

Furthermore, the wage must be equal to the marginal product of labour, MPN

$$\begin{aligned}
w &= \frac{\partial Y}{\partial K} = AF\left(\frac{K}{AL}, 1\right) - AL \cdot \frac{K}{AL^2} F_1\left(\frac{K}{AL}, 1\right) \\
&= Af(\tilde{k}) - A\tilde{k}f'(\tilde{k})
\end{aligned}$$